

A CONTINUOUS TIME STOCHASTIC MODEL FOR BIOLOGICAL NEURAL NETS

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ABSTRACT. We propose a new stochastic model for biological neural nets which is a continuous time version of the model proposed by Galves and Löcherbach in [1]. We also show how to computationally simulate such model for easy neuron potential decays and probability functions and characterize when the model has a finite time of death almost surely.

1. INTRODUCTION

Based on a stochastic model for biological neural nets proposed by Galves and Löcherbach in [1] and on a specific algorithm for simulating such model, we propose a model on continuous time. Formally, the model is actually still discrete but generates times of neuron discharges which could be interpreted as a boolean stochastic process on continuous time.

One common approach to define continuous time models is to consider stochastic differential equations, but the approach considered here is more of an algorithmic oriented approach and throughout this work we focused on producing a model that can be efficiently simulated.

Since the nature of the model is algorithmic, we choose to first present the model informally with a more intuitive and constructive approach and only formalize it on the final section.

Also in the final section we present a characterization of when the system dies almost surely, i.e., when there is a time $t_0 \in \mathbb{R}_+$ such that there are no neuron discharges after t_0 .

2. DISCRETE TIME MODEL

In this section we present the discrete time model proposed by Galves and Löcherbach [1] and algorithms to simulate it. Such algorithms will provide a useful insight on how to define an analogous continuous time model.

For simplicity, we will consider in this section the model with a finite set of neurons and with no potential decay, but we stress that Galves and Löcherbach did study the case of a countable set of neurons and with potential decays much more general than the ones we will consider in the continuous time model.

2.1. The model. We first assume fixed a finite set I of neurons, a family of potential probability functions $(\varphi_i)_{i \in I}$. For every $i \in I$, the function $\varphi_i: \mathbb{R} \rightarrow [0, 1]$ maps the potential of neuron i to the probability that it will fire at that time. For every $i, j \in I$, we also assume fixed $W_{i \rightarrow j} \in \mathbb{R}$, which gives the influence that a discharge of neuron i has on neuron j .

The model consists of a stochastic chain $(X_t)_{t \in \mathbb{Z}}$ taking values in $\{0, 1\}^I$ and an auxiliary chain $(U_t)_{t \in \mathbb{Z}}$ taking values in \mathbb{R}^I . For each neuron $i \in I$, and each time $t \in \mathbb{Z}$, the value $X_t(i)$ will be 1 if neuron i fires at time t and 0 otherwise. Furthermore, each neuron $i \in I$ will have an internal potential at time $t \in \mathbb{Z}$ given by $U_t(i)$.

We define now the filtration

$$\mathcal{F}_t = \sigma[\{X_s : s \in \mathbb{Z}, s \leq t\}], \quad t \in \mathbb{Z},$$

and, for every time $t \in \mathbb{Z}$ and neuron $i \in I$, we let

$$L_t^i = \sup\{s \leq t : X_s(i) = 1\}$$

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be the F_t -measurable time of the last discharge of neuron i up to time t . The potential of neuron i at time t is then defined as the F_t -measurable random variable

$$U_t(i) = \max \left\{ \sum_{s=L_t^i+1}^t \sum_{j \in J} W_{j \rightarrow i} X_s(j), 0 \right\}.$$

The dynamics of the process is then the following: at time $t+1$, the probability of neuron discharges are independent conditionally on the whole past, i.e., we have

$$\mathbb{P}(\forall i \in I, X_{t+1}(i) = a_i \mid \mathcal{F}_t) = \prod_{i \in I} \mathbb{P}(X_{t+1}(i) = a_i \mid \mathcal{F}_t),$$

furthermore, the probability of having a discharge from neuron $i \in I$ at time $t+1$ is given by

$$\mathbb{P}(X_{t+1}(i) = 1 \mid \mathcal{F}_t) = \varphi_i(U_t(i)).$$

Note that although we define the chain depending on a great part of its past, the probabilities of discharge at time $t+1$ can be calculated knowing only the potentials at time t and potentials at time $t+1$ can be calculated knowing only the potentials at time t and the discharges at time t .

2.2. Algorithms. We now present two different simple algorithms to simulate the discrete model from an initial state of potentials U_0 .

2.2.1. Single-step algorithm. The single-step algorithm consists of simulating the evolution of the process step-by-step. Given potentials U_t at time t , we sample $|I|$ independent uniform random variables $(V_t(i))_{i \in I}$ over $[0, 1)$ and let

$$X_{t+1}(i) = \mathbb{1}_{\{V_t(i) < \varphi_i(U_t(i))\}}.$$

Furthermore, we update the potentials by letting

$$\begin{aligned} U_{t+1}(i) &= \begin{cases} 0, & \text{if } X_{t+1}(i) = 1; \\ \max \left\{ U_t(i) + \sum_{j \in I} W_{j \rightarrow i} X_{t+1}(j), 0 \right\}, & \text{if } X_{t+1}(i) = 0; \end{cases} \\ &= (1 - X_{t+1}(i)) \max \left\{ U_t(i) + \sum_{j \in I} W_{j \rightarrow i} X_{t+1}(j), 0 \right\} \end{aligned}$$

The simulation consists of repeating this process inductively from $t = 0$.

Is easy to see that this simulation indeed produces the correct stochastic process.

2.2.2. Multi-step algorithm. The multi-step algorithm consists of skipping steps of the single-step algorithm where no neuron discharges occur. To do this, we first observe that in the absence of other neuron discharges, the time we have to wait from time t_0 to see a discharge from neuron i has geometric distribution of parameter $\varphi_i(U_{t_0}(i))$.

Given potentials U_t at time t , we sample $|I|$ independent random variables $(T_t(i))_{i \in I}$ with $T_t(i)$ having geometric distribution with parameter $\varphi_i(U_t(i))$ for every $i \in I$ and let $T_t = \min\{T_t(i) : i \in I\}$ and $D_t = \{i \in I : T_t(i) = T_t\}$.

The variable T_t gives how much time we have to wait to see a neuron firing, so we skip this amount of time by letting $U_{t+s}(i) = U_t(i)$ and $X_{t+s}(i) = 0$ for every $0 < s < T_t$ and $i \in I$ and then let

$$X_{t+T_t}(i) = \mathbb{1}_{\{i \in D_t\}}.$$

Furthermore, we update the potentials by letting

$$U_{t+T_t}(i) = \begin{cases} 0, & \text{if } X_{t+T_t}(i) = 1; \\ \max \left\{ U_t(i) + \sum_{j \in I} W_{j \rightarrow i} X_{t+T_t}(j), 0 \right\}, & \text{if } X_{t+T_t}(i) = 0; \end{cases}$$

$$= (1 - X_{t+T_t}(i)) \max \left\{ U_t(i) + \sum_{j \in I} W_{j \rightarrow i} X_{t+T_t}(j), 0 \right\}.$$

Proposition 2.1. The multi-step algorithm correctly simulates the discrete time stochastic process.

Proof. Throughout this proof, we denote the random variables produced by the single-step algorithm by X' and U' .

For every time t , let

$$T'_t(i) = \inf\{s > t : X'_s(i) = 1\};$$

$$T'_t = \min\{T'_t(i) : i \in I\};$$

$$D'_t = \{i \in I : X'_{T'_t}(i) = 1\}.$$

Note that it is enough to prove that the random variables (T'_0, D'_0) and (T_0, D_0) have the same law (because after times T_0 and T'_0 , the algorithms depend only on U_{T_0} and $U'_{T'_0}$ respectively).

Observe now that for every $t_0 > 0$ and every $A_0 \subset I$ not-empty, we have that $(T'_0, D'_0) = (t_0, A_0)$ if and only if

$$X'_t(i) = 0, \quad \forall i \in I, \forall t < t_0;$$

$$X'_{t_0}(i) = 0, \quad \forall i \in I \setminus A_0;$$

$$X'_{t_0}(i) = 1, \quad \forall i \in A_0;$$

hence we have

$$\begin{aligned} \mathbb{P}((T'_0, D'_0) = (t_0, A_0)) &= \prod_{t=1}^{t_0-1} \prod_{i \in I} (1 - \varphi_i(U_0(i))) \prod_{i \in I \setminus A_0} (1 - \varphi_i(U_0(i))) \prod_{i \in A_0} \varphi_i(U_0(i)) \\ &= \prod_{i \in I \setminus A_0} (1 - \varphi_i(U_0(i)))^{t_0} \prod_{i \in A_0} (1 - \varphi_i(U_0(i)))^{t_0-1} \varphi_i(U_0(i)), \end{aligned}$$

because the potentials of the neurons remain the same if there is no neuron discharge.

Therefore we have

$$\begin{aligned} \mathbb{P}((T'_0, D'_0) = (t_0, A_0)) &= \prod_{i \in I \setminus A_0} \mathbb{P}(T_0(i) > t_0) \prod_{i \in A_0} \mathbb{P}(T_0(i) = t_0) \\ &= \mathbb{P}(\forall i \in I \setminus A_0, T_0(i) > t_0 \text{ and } \forall i \in A_0, T_0(i) = t_0) \\ &= \mathbb{P}((T_0, D_0) = (t_0, A_0)), \end{aligned}$$

because of the independence of the random variables $(T_0(i))_{i \in I}$.

Thus we have proved that (T'_0, D'_0) and (T_0, D_0) have the same law. ■

3. CONTINUOUS TIME MODEL

We introduce now the continuous time stochastic process. The idea is to define the process from the multi-step algorithm presented in the previous section.

We suppose that an initial potential $(U_0(i))_{i \in I}$ is given. Naturally, the difference from this model to the last is that the chains $(X_t)_{t \in \mathbb{R}_+}$ and $(U_t)_{t \in \mathbb{R}_+}$ are now indexed by continuous time.

Now we would like to add decay on the potentials of the neurons so we suppose that, for every $i \in I$, we are given a function $V_i: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that will give how the potential of neuron i decays in the absence of neuron discharges: the value $V_i(u, s)$ will be how much

potential is left after a total time s has passed if the neuron started with potential u (naturally, we suppose $V_i(u, 0) = u$ for every $u > 0$).

The dynamics of the process is then defined by the following evolution algorithm.

Given potentials U_t at time t , we sample $|I|$ independent random variables $(T_t(i))_{i \in I}$ with $T_t(i)$ having a certain distribution (which will depend on the choice of V_i and which we will study later) for every $i \in I$ and let $T_t = \min\{T_t(i) : i \in I\}$ and $D_t = \{i \in I : T_t(i) = T_t\}$.

Then we let $U_{t+s}(i) = V_i(U_t(i), s)$ and $X_{t+s}(i) = 0$ for every $0 < s < T_t$ and $i \in I$ and let

$$X_{t+T_t}(i) = \mathbb{1}_{\{i \in D_t\}}.$$

Furthermore, we update the potentials by letting

$$\begin{aligned} U_{t+T_t}(i) &= \begin{cases} 0, & \text{if } X_{t+T_t}(i) = 1; \\ \max \left\{ V_i(U_t(i), T_t) + \sum_{j \in I} W_{j \rightarrow i} X_{t+T_t}(j), 0 \right\}, & \text{if } X_{t+T_t}(i) = 0; \end{cases} \\ &= (1 - X_{t+T_t}(i)) \max \left\{ V_i(U_t(i), T_t) + \sum_{j \in I} W_{j \rightarrow i} X_{t+T_t}(j), 0 \right\}. \end{aligned}$$

3.1. Distribution of wait time. In this section, we will study what kind of distribution we should put in the random variables $T_t(i)$'s. We assume that t and i are fixed, and we will denote $T_t(i)$ by simply T until the end of this section (Section 3).

In the multi-step algorithm of the discrete time model, this variable had geometric distribution with parameter $\lambda = \varphi(U)$ (here we also are dropping the notation by letting $\varphi \equiv \varphi_i$ and $U = U_t(i)$). This means that, for every $k \in \mathbb{N}^*$, we had

$$\mathbb{P}(T = k) = (1 - \lambda)^{k-1} \lambda = \mathbb{P}(T < k) \lambda,$$

so the natural analogous distribution in continuous time is to consider a probability density function ρ satisfying

$$\rho(t) = \left(1 - \int_0^t \rho(s) ds \right) \lambda.$$

Now we have that ρ must be a solution of the following differential equation

$$\rho'(t) = -\lambda \rho(t),$$

hence it is of the form

$$\rho(t) = C e^{-\lambda t},$$

where $C \in \mathbb{R}$ is a constant.

Substituting back in the original equation, we have

$$\int_0^t \rho(s) ds = 1 - \frac{\rho(t)}{\lambda} = 1 - \frac{C e^{-\lambda t}}{\lambda},$$

and, using the condition $\int_0^0 \rho(s) ds = 0$, we deduce that $C = \lambda$, hence we obtain the exponential distribution of parameter λ , which was expected.

But we are now interested in allowing the parameter λ to vary in time. This means that ρ should satisfy

$$\rho(t) = \left(1 - \int_0^t \rho(s) ds \right) \lambda(t),$$

which leads to the following differential equation

$$\frac{\rho'(t) \lambda(t) - \rho(t) \lambda'(t)}{\lambda^2(t)} = \rho(t),$$

whose solutions are of the form

$$\rho(t) = C\lambda(t) \exp\left(-\int_0^t \lambda(s)ds\right),$$

where $C \in \mathbb{R}$ is a constant.

Substituting back in the original equation, we have

$$\int_0^t \rho(s)ds = 1 - \frac{\rho(t)}{\lambda(t)} = 1 - C \exp\left(-\int_0^t \lambda(s)ds\right),$$

and again, using the condition $\int_0^0 \rho(s)ds$, we deduce that $C = 1$.

Note now that $\int_0^{+\infty} \rho(s)ds = 1$ if and only if $\int_0^{+\infty} \lambda(s)ds = +\infty$. The interpretation of this is that if $\lambda(t)$ decreases too fast, then there is a non-zero probability that the neuron will never fire.

Now define the function

$$F: [0, +\infty] \longrightarrow [0, 1]$$

$$t \longmapsto \begin{cases} \int_0^t \rho(s)ds, & \text{if } t < +\infty; \\ 1, & \text{if } t = +\infty; \end{cases}$$

which is the analogous of a cumulative distribution function with the difference that the random variable can take the value $+\infty$.

Therefore, the distribution of T is given by

$$\begin{cases} \mathbb{P}(T < t) = F(t), & \forall t < +\infty; \\ \mathbb{P}(T = +\infty) = F(+\infty) - \lim_{s \rightarrow +\infty} F(s) = 1 - \int_0^{+\infty} \rho(s)ds. \end{cases}$$

Now we let $\lambda(t) = \varphi(V(t))$ (again dropping the notation by letting $V(s) = V_i(U(t), s)$).

In the next sections, we will study decay laws that satisfy the following differential equation

$$V'(t) = -\mu V^\gamma(t),$$

where $\gamma \geq 1$ is a fixed constant.

Note that in this particular case we have

$$\int_0^t \lambda(s)ds = \int_0^t \varphi(V(s))ds = \int_0^t \frac{\varphi(V(s))V'(s)}{-\mu V^\gamma(s)}ds = \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\varphi(v)}{v^\gamma}dv.$$

and $\lim_{t \rightarrow +\infty} V(t) = 0$.

Furthermore, we are interested in the following potential probability functions

- (1) Exponential: $\varphi^{(\text{exp})}(u) = 1 - e^{-\beta u}$, for $\beta > 0$ fixed;
- (2) Rational: $\varphi^{(r)}(u) = u^r/(u^r + \beta)$, for $r \in \mathbb{N}^*$ and $\beta > 0$ fixed;
- (3) Monomial: $\varphi_r(u) = \beta u^r$, for $r \in \mathbb{N}^*$ and $\beta > 0$ fixed;

3.2. Exponential decay. In the case $\gamma = 1$, we have the following differential equation

$$V'(t) = -\mu V(t),$$

whose solutions are of the form

$$V(t) = V(0)e^{-\mu t},$$

which is the exponential decay typical of radioactive decay models.

For this type of decay, we can prove a very interesting property.

Proposition 3.1. If $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function with $g(u) = 0$ if and only if $u = 0$ and continuously differentiable in a neighbourhood of 0 and $r > 0$ is a positive real number, then taking $V(t) = V(0)e^{-\mu t}$ and $\varphi \equiv g^r$ yields $\mathbb{P}(T = +\infty) > 0$.

Proof. Since g is continuously differentiable in a neighbourhood of 0 and $g(0) = 0$, for every $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$g(t) \leq (g'(0) + \varepsilon)t,$$

for every $0 \leq t < t_0$, hence

$$\varphi(t) \leq (g'(0) + \varepsilon)^r t^r.$$

This means that

$$\begin{aligned} \int_0^{t_0} \frac{\varphi(v)}{v} dv &\leq \int_0^{t_0} \frac{(g'(0) + \varepsilon)^r v^r}{v} dv \\ &= (g'(0) + \varepsilon)^r \frac{t_0^r}{r} \Big|_0^{t_0} = (g'(0) + \varepsilon)^r \frac{t_0^r}{r} < +\infty, \end{aligned}$$

where the last two equalities follow from the fact that $r > 0$.

Hence

$$\begin{aligned} \int_0^{+\infty} \lambda(s) ds &= \frac{1}{\mu} \int_0^{V(0)} \frac{\varphi(v)}{v} dv \\ &= \frac{1}{\mu} \left(\int_0^{t_0} \frac{\varphi(v)}{v} dv + \int_{t_0}^{V(0)} \frac{\varphi(v)}{v} dv \right) \\ &< +\infty \end{aligned}$$

and, therefore

$$\mathbb{P}(T < +\infty) = 1 - \exp\left(-\int_0^{+\infty} \lambda(s) ds\right) < 1.$$

■

Corollary 3.2. Suppose that, for every $i \in I$, we are given a positive real number $r_i > 0$ and a function $g_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $g_i(0) = 0$ and such that g_i is continuous and continuously differentiable in a neighbourhood of 0.

Then the continuous time model with $\varphi_i \equiv g_i^r$ for every $i \in I$ and exponential decay has non-null probability of never firing, i.e., we have

$$\mathbb{P}(\forall t > 0, \forall i \in I, X_t(i) = 0) > 0.$$

Remark. Clearly from the proof of Proposition 3.1, we only need Lipschitz condition on a neighbourhood of 0 and locally boundedness of g , but we chose not to state the proposition in the more general form for simplicity.

3.2.1. Pratical sampling. Although the continuous time algorithm is well-defined, we are interested in actually implementing it with a reasonable efficiency, so in this section we show how one can sample the random variable T of the algorithm from a uniform random variable Z over $[0, 1)$.

For the case of the rational potential $\varphi^{(r)}$, note that, for every $t < +\infty$, we have

$$\begin{aligned} \int_0^t \lambda(s) ds &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\varphi^{(r)}(v)}{v} dv \\ &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{v^{r-1}}{v^r + \beta} dv \\ &= \frac{1}{\mu} \frac{\ln(v^r + \beta)}{r} \Big|_{V(t)}^{V(0)} \\ &= \frac{1}{r\mu} \ln \frac{V(0)^r + \beta}{V(t)^r + \beta} \xrightarrow{t \rightarrow +\infty} \frac{1}{r\mu} \ln \frac{V(0)^r + \beta}{\beta}. \end{aligned}$$

Hence, for every $t < +\infty$, we have

$$\begin{aligned} F(t) &= 1 - \exp\left(-\int_0^t \lambda(s)ds\right) = 1 - \left(\frac{V(0)^r + \beta}{V(t)^r + \beta}\right)^{-1/(r\mu)} \\ &= 1 - \left(\frac{V(0)^r + \beta}{V(0)^r e^{-\mu r t} + \beta}\right)^{-1/(r\mu)} \xrightarrow{t \rightarrow +\infty} 1 - \left(\frac{V(0)^r + \beta}{\beta}\right)^{-1/(r\mu)}. \end{aligned}$$

With a simple calculation, for every $t < +\infty$, we have

$$t = -\frac{1}{r\mu} \ln \frac{(V(0)^r + \beta)(1 - F(t))^{r\mu} - \beta}{V(0)^r},$$

hence defining

$$\begin{aligned} G: [0, 1) &\longrightarrow [0, +\infty] \\ \xi &\longmapsto \begin{cases} -\frac{1}{r\mu} \ln \frac{(V(0)^r + \beta)(1 - \xi)^{r\mu} - \beta}{V(0)^r}, & \text{if } \xi < 1 - \left(\frac{V(0)^r + \beta}{\beta}\right)^{-1/(r\mu)}; \\ +\infty, & \text{otherwise;} \end{cases} \end{aligned}$$

we have $\mathbb{P}(G(Z) < t) = F(t)$ for every $t < +\infty$ and $\mathbb{P}(G(Z) < +\infty) = \lim_{t \rightarrow +\infty} F(t)$, thus giving an easy way of sampling T from Z .

Remark. Note that taking the limit $V(0) \rightarrow +\infty$, we have that $G(\xi) = -\ln(1 - \xi)$ for every $\xi \in [0, 1)$, this means that if $V(0)$ is arbitrarily large, the wait time has exponential distribution with parameter 1 ($\xi = 1 - e^{-G(\xi)}$).

The case of the exponential potential $\varphi^{(\text{exp})}$ also gives an invertible $F(t)$, but the inverse cannot be expressed in terms of essential functions, which makes it very unpractical to compute.

We will now consider the monomial potential φ_r . Note that, for every $t < +\infty$, we have

$$\begin{aligned} \int_0^t \lambda(s)ds &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\varphi_r(v)}{v} dv \\ &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \beta v^{r-1} dv \\ &= \frac{\beta(V(0)^r - V(t)^r)}{r\mu} \\ &\xrightarrow{t \rightarrow +\infty} \frac{\beta V(0)^r}{r\mu} \end{aligned}$$

Hence, for every $t < +\infty$, we have

$$\begin{aligned} F(t) &= 1 - \exp\left(-\int_0^t \lambda(s)ds\right) = 1 - \exp\left(-\frac{\beta(V(0)^r - V(t)^r)}{r\mu}\right) \\ &= 1 - \exp\left(-\frac{\beta V(0)^r (1 - e^{-\mu r t})}{r\mu}\right) \xrightarrow{t \rightarrow +\infty} 1 - \exp\left(-\frac{\beta V(0)^r}{r\mu}\right). \end{aligned}$$

With a simple calculation, for every $t < +\infty$, we have

$$t = -\frac{1}{\mu r} \ln \left(1 - \frac{r\mu \ln(1 - F(t))}{\beta V(0)^r}\right),$$

hence defining

$$\begin{aligned} G: [0, 1) &\longrightarrow [0, +\infty] \\ \xi &\longmapsto \begin{cases} -\frac{1}{\mu r} \ln \left(1 - \frac{r\mu \ln(1 - \xi)}{\beta V(0)^r}\right), & \text{if } \xi < 1 - \exp\left(-\frac{\beta V(0)^r}{r\mu}\right); \\ +\infty, & \text{otherwise;} \end{cases} \end{aligned}$$

we have $\mathbb{P}(G(Z) < t) = F(t)$ for every $t < +\infty$ and $\mathbb{P}(G(Z) < +\infty) = \lim_{t \rightarrow +\infty} F(t)$, thus giving an easy way of sampling T from Z .

Remark. Note that taking the limit $V(0) \rightarrow +\infty$, we have that $G(\xi) = 0$ for every $\xi \in [0, 1]$, this means that if $V(0)$ is arbitrarily large, the neuron fires immediately. This behaviour is typical of potential functions such that $\lim_{u \rightarrow +\infty} \varphi(u) = +\infty$.

3.3. Other decays. Now we will cover the case where $V(t)$ satisfies the following differential equation

$$V'(t) = -\mu V(t)^\gamma,$$

with $\gamma > 1$ a fixed constant.

The solutions of this differential equation are of the form

$$V(t) = (V(0)^{1-\gamma} + (\gamma - 1)\mu t)^{1/(1-\gamma)},$$

which in the particular case when $\gamma = 2$ gives the reciprocal decay

$$V(t) = \frac{1}{\mu t + V(0)^{-1}}.$$

And in this cases we have a proposition analogous to Proposition 3.1.

Proposition 3.3. Suppose $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function vanishing only at 0 and of class $\mathcal{C}^{[\gamma]}$ on a neighbourhood of 0 and

$$\frac{d^k g}{dt^k}(0) = 0,$$

for every $k = 0, 1, \dots, [\gamma] - 1$.

Then for every $r > (\gamma - 1)/[\gamma]$, taking $V(t) = (V(0)^{1-\gamma} + (\gamma - 1)\mu t)^{1/(1-\gamma)}$ and $\varphi \equiv g^r$ yields $\mathbb{P}(T = +\infty) > 0$.

Proof. Note that $[\gamma]r - \gamma + 1 > 0$. Since g is of class $\mathcal{C}^{[\gamma]}$ on a neighbourhood of 0 and all the $[\gamma] - 1$ first derivatives of g are null on 0, we have that for every $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$g(t) \leq (g^{([\gamma])}(0) + \varepsilon) \frac{t^{[\gamma]}}{[\gamma]!},$$

for every $0 < t < t_0$, where $g^{([\gamma])}$ is the $[\gamma]$ -th derivative of g . Hence we have

$$\varphi(t) \leq (g^{([\gamma])}(0) + \varepsilon)^r \frac{t^{[\gamma]r}}{[\gamma]!^r}.$$

This means that

$$\begin{aligned} \int_0^{t_0} \frac{\varphi(v)}{v^\gamma} dv &\leq \int_0^{t_0} \frac{(g^{([\gamma])}(0) + \varepsilon)^r t^{[\gamma]r}}{[\gamma]!^r v^\gamma} dv \\ &= \left(\frac{g^{([\gamma])}(0) + \varepsilon}{[\gamma]!} \right)^r \frac{t^{[\gamma]r - \gamma + 1}}{[\gamma]r - \gamma + 1} \Big|_0^{t_0} \\ &= \left(\frac{g^{([\gamma])}(0) + \varepsilon}{[\gamma]!} \right)^r \frac{t_0^{[\gamma]r}}{[\gamma]!^r}, \end{aligned}$$

where the last two equalities follow from the fact that $[\gamma]r - \gamma + 1 > 0$.

Hence

$$\begin{aligned}\int_0^{+\infty} \lambda(s) ds &= \frac{1}{\mu} \int_0^{V(0)} \frac{\varphi(v)}{v^\gamma} dv \\ &= \frac{1}{\mu} \left(\int_0^{t_0} \frac{\varphi(v)}{v^\gamma} dv + \int_{t_0}^{V(0)} \frac{\varphi(v)}{v^\gamma} dv \right) \\ &< +\infty\end{aligned}$$

and, therefore

$$\mathbb{P}(T < +\infty) = 1 - \exp\left(-\int_0^{+\infty} \lambda(s) ds\right) < 1.$$

■

Corollary 3.4. Suppose that, for every $i \in I$, we are given a real number $r_i > (\gamma - 1)/\lfloor \gamma \rfloor$ function $g_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that g_i is continuous and continuously differentiable $\lfloor \gamma \rfloor$ times in a neighbourhood of 0 and the first $\lfloor \gamma \rfloor - 1$ derivatives of g and g itself are all null on 0.

Then the continuous time model with $\varphi_i \equiv g_i^r$ for every $i \in I$ and decay with parameter γ has non-null probability of never firing, i.e., we have

$$\mathbb{P}(\forall t > 0, \forall i \in I, X_t(i) = 0) > 0.$$

Remark. Clearly from the proof of Proposition 3.3, we only need $\lfloor \gamma \rfloor$ -Hölder condition on a neighbourhood of 0 and locally boundedness of g , but again we chose not to state the proposition in the more general form for simplicity.

3.3.1. Practical sampling. Again we show that for some potentials we can sample the random variable T of the algorithm from a uniform random variable Z over $[0, 1)$.

For simplicity, we will only show the cases with $\gamma = 2$.

For the rational potential $\varphi^{(r)}$ with $r = 1$, note that, for every $t < +\infty$, we have

$$\begin{aligned}\int_0^t \lambda(s) ds &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\varphi^{(1)}(v)}{v^2} dv \\ &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{1}{v(v + \beta)} dv \\ &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \left(\frac{\beta^{-1}}{v} - \frac{\beta^{-1}}{v + \beta} \right) dv \\ &= \frac{1}{\beta\mu} \ln \frac{v}{v + \beta} \Big|_{V(t)}^{V(0)} \\ &= \frac{1}{\beta\mu} \ln \frac{V(0)(V(t) + \beta)}{V(t)(V(0) + \beta)} \xrightarrow{t \rightarrow +\infty} +\infty.\end{aligned}$$

Hence, for every $t < +\infty$, we have

$$\begin{aligned}F(t) &= 1 - \exp\left(-\int_0^t \lambda(s) ds\right) = 1 - \left(\frac{V(0)(V(t) + \beta)}{V(t)(V(0) + \beta)}\right)^{-1/(\beta\mu)} \\ &= 1 - \left(\frac{V(0)((\mu t + V(0)^{-1})^{-1} + \beta)}{(\mu t + V(0)^{-1})^{-1}(V(0) + \beta)}\right)^{-1/(\beta\mu)} \xrightarrow{t \rightarrow +\infty} 1.\end{aligned}$$

With a simple calculation, for every $t < +\infty$, we have

$$t = \frac{(1 - F(t))^{-\beta\mu} - 1}{\mu} \left(\frac{1}{V(0)} + \frac{1}{\beta} \right),$$

hence defining

$$\begin{aligned} G: [0, 1) &\longrightarrow [0, +\infty] \\ \xi &\longmapsto \frac{(1 - \xi)^{-\beta\mu} - 1}{\mu} \left(\frac{1}{V(0)} + \frac{1}{\beta} \right) \end{aligned}$$

we have $\mathbb{P}(G(Z) < t) = F(t)$ for every $t < +\infty$, thus giving an easy way of sampling T from Z .

Remark. Note that in this case G ommits the value $+\infty$, which means that the neuron will always fire in finite time.

Note also that taking the limit $V(0) \rightarrow +\infty$, we have that $G(\xi) = ((1 - \xi)^{-\beta\mu} - 1)/(\beta\mu)$ for every $\xi \in [0, 1)$.

For the rational potential $\varphi^{(r)}$ with $r = 2$, note that, for every $t < +\infty$, we have

$$\begin{aligned} \int_0^t \lambda(s) ds &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\varphi^{(2)}(v)}{v^2} dv \\ &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{1}{v^2 + \beta} dv \\ &= \frac{1}{\mu} \frac{1}{\sqrt{\beta}} \arctan \frac{v}{\sqrt{\beta}} \Big|_{V(t)}^{V(0)} \\ &= \frac{1}{\sqrt{\beta}\mu} \left(\arctan \frac{V(0)}{\sqrt{\beta}} - \arctan \frac{V(t)}{\sqrt{\beta}} \right) \xrightarrow{t \rightarrow +\infty} \frac{1}{\sqrt{\beta}\mu} \arctan \frac{V(0)}{\sqrt{\beta}}. \end{aligned}$$

Hence, for every $t < +\infty$, we have

$$\begin{aligned} F(t) &= 1 - \exp \left(- \int_0^t \lambda(s) ds \right) = 1 - \exp \left(- \frac{1}{\sqrt{\beta}\mu} \left(\arctan \frac{V(0)}{\sqrt{\beta}} - \arctan \frac{V(t)}{\sqrt{\beta}} \right) \right) \\ &= 1 - \exp \left(- \frac{1}{\sqrt{\beta}\mu} \left(\arctan \frac{V(0)}{\sqrt{\beta}} - \arctan \frac{(\mu t + V(0)^{-1})^{-1}}{\sqrt{\beta}} \right) \right) \\ &\xrightarrow{t \rightarrow +\infty} 1 - \exp \left(- \frac{1}{\sqrt{\beta}\mu} \arctan \frac{V(0)}{\sqrt{\beta}} \right). \end{aligned}$$

Again with a simple calculation, for every $t < +\infty$, we have

$$t = \frac{1}{\mu} \left(\frac{1}{\sqrt{\beta}} \cot \left(\arctan \frac{V(0)}{\sqrt{\beta}} + \mu\sqrt{\beta} \ln(1 - F(t)) \right) - \frac{1}{V(0)} \right),$$

hence defining

$$\begin{aligned} G: [0, 1) &\longrightarrow [0, +\infty] \\ \xi &\longmapsto \begin{cases} \frac{1}{\mu} \left(\frac{1}{\sqrt{\beta}} \cot \left(\arctan \frac{V(0)}{\sqrt{\beta}} + \mu\sqrt{\beta} \ln(1 - \xi) \right) - \frac{1}{V(0)} \right), & \text{if } \xi < L; \\ +\infty, & \text{otherwise;} \end{cases} \end{aligned}$$

where

$$L = 1 - \exp \left(- \frac{1}{\sqrt{\beta}\mu} \arctan \frac{V(0)}{\sqrt{\beta}} \right),$$

we have $\mathbb{P}(G(Z) < t) = F(t)$ for every $t < +\infty$ and $\mathbb{P}(G(Z) < +\infty) = \lim_{t \rightarrow +\infty} F(t)$, thus giving an easy way of sampling T from Z .

Remark. Note that taking the limit $V(0) \rightarrow +\infty$, we have that

$$G(\xi) = \begin{cases} \frac{1}{\sqrt{\beta}\mu} \tan \left(-\mu\sqrt{\beta} \ln(1 - \xi) \right), & \text{if } \xi < 1 - \exp \left(-\frac{\pi}{2\sqrt{\beta}\mu} \right); \\ +\infty, & \text{otherwise;} \end{cases}$$

for every $\xi \in [0, 1)$, this means that even if $V(0)$ is arbitrarily large, there is a probability of at least $\exp(-\pi/(2\sqrt{\beta}\mu))$ that the neuron will not fire.

Now, for the monomial decay φ_r with $r = 1$, for every $t < +\infty$, we have

$$\begin{aligned}\int_0^t \lambda(s) ds &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\varphi_1(v)}{v^2} dv \\ &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\beta}{v} dv \\ &= \frac{\beta}{\mu} \ln \frac{V(0)}{V(t)} \\ &\xrightarrow{t \rightarrow +\infty} +\infty.\end{aligned}$$

Hence, for every $t < +\infty$, we have

$$\begin{aligned}F(t) &= 1 - \exp\left(-\int_0^t \lambda(s) ds\right) = 1 - \left(\frac{V(0)}{V(t)}\right)^{-\beta/\mu} \\ &= 1 - (1 + V(0)\mu t)^{-\beta/\mu} \xrightarrow{t \rightarrow +\infty} 1.\end{aligned}$$

With a simple calculation, for every $t < +\infty$, we have

$$t = \frac{(1 - F(t))^{-\mu/\beta} - 1}{\mu V(0)},$$

hence defining

$$\begin{aligned}G: [0, 1) &\longrightarrow [0, +\infty] \\ \xi &\longmapsto \frac{(1 - \xi)^{-\mu/\beta} - 1}{\mu V(0)},\end{aligned}$$

we have $\mathbb{P}(G(Z) < t) = F(t)$ for every $t < +\infty$, thus giving an easy way of sampling T from Z .

Remark. Note that taking the limit $V(0) \rightarrow +\infty$, we have that $G(\xi) = 0$, this means that if $V(0)$ is arbitrarily large, the neuron fires immediately.

Last, but not least, we consider the monomial potential φ_r with $r \geq 2$. Note that, for every $t < +\infty$, we have

$$\begin{aligned}\int_0^t \lambda(s) ds &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \frac{\varphi_r(v)}{v^2} dv \\ &= \frac{1}{\mu} \int_{V(t)}^{V(0)} \beta v^{r-2} dv \\ &= \frac{\beta}{(r-1)\mu} (V(0)^{r-1} - V(t)^{r-1}) \\ &\xrightarrow{t \rightarrow +\infty} +\infty,\end{aligned}$$

which coincides with the case φ_{r-1} with exponential decay (with a different β).

We hope that these examples were enough to illustrate that by changing the potential decay law and the potential probability functions we can obtain very distinct wait time distributions.

4. FORMALIZATION

We will now formalize the model.

Suppose I is a finite set and $(V_i)_{i \in I} \in \left(\mathbb{R}_+^{\mathbb{R}_+ \times \mathbb{R}_+}\right)^I$ and $(\varphi_i)_{i \in I} \in \left(\mathbb{R}_+^{\mathbb{R}_+}\right)^I$ are families of functions such that

1. For every $u \in \mathbb{R}_+$ and every $i \in I$, we have $V_i(u, 0) = u$;
2. For every $t \in \mathbb{R}_+$ and every $i \in I$, the function $V_i(\cdot, t)$ is non-decreasing;
3. For every $u \in \mathbb{R}_+$ and every $i \in I$, the function $V_i(u, \cdot)$ is non-increasing;

4. For every $u \in \mathbb{R}_+$, every $i \in I$ and every $t, t' \in \mathbb{R}_+$, we have

$$V_i(u, t + t') = V_i(V_i(u, t), t');$$

5. For every $i \in I$, the function φ_i is non-decreasing;

6. For every $i \in I$, the function $\varphi_i \circ V_i$ is Borel-measurable (on both coordinates);

Suppose furthermore that $W_{i \rightarrow j} \in \mathbb{R}^{I \times I}$ is a matrix of neuron influences and that $(Z_{i,n})_{i \in I, n \in \mathbb{N}}$ is a family of independent identically distributed uniform random variables over $[0, 1)$.

Suppose finally that $(a(i))_{i \in I} \in \mathbb{R}_+^I$ is a vector of initial potentials.

We now define sequences $(T_n(i))_{n \in \mathbb{N}, i \in I}$, $(X_n(i))_{n \in \mathbb{N}, i \in I}$ and $(U_n(i))_{n \in \mathbb{N}, i \in I}$ of random variables inductively as follows.

- i. Let $U_0(i) = a(i)$ for every $i \in I$;
- ii. For every $n \in \mathbb{N}$, let

$$T_n(i) = \sup \left\{ t \in [0, +\infty) : 1 - \exp \left(- \int_0^t \varphi_i(V_i(U_n(i), s)) dm(s) \right) \leq Z_{i,n} \right\};$$

- iii. For every $n \in \mathbb{N}$, let $T_n = \min\{T_n(i) : i \in I\}$;

If $T_n = +\infty$, let $U_m(i) = \lim_{t \rightarrow +\infty} V_i(U_n(i), t)$, $T_m(i) = T_m = +\infty$ and $X_m(i) = 0$ for every $i \in I$ and every $m > n$ and stop the induction;

- iv. For every $n \in \mathbb{N}$, let $X_n(i) = \mathbb{1}_{\{T_n(i) = T_n\}}$;
- v. For every $n \in \mathbb{N}$, let

$$U_{n+1}(i) = (1 - X_n(i)) \max \left\{ \lim_{t \rightarrow T_n^-} V_i(U_n(i), t) + \sum_{j \in I} W_{j \rightarrow i} X_n(j), 0 \right\}.$$

Finally, define, for every $n \in \mathbb{N}$, define $T'_n = \sum_{n \in \mathbb{N}} T_n$.

Observation. The condition

$$V_i(u, t + t') = V_i(V_i(u, t), t')$$

might seem unnatural at first, but it is an underlying condition of the algorithm because it says that interrupting the potential evolution at any time t and restarting the evolution from the value $V_i(u, t)$ yields the same result of not interrupting. Such interruptions are made in the algorithm whenever a neuron fires and this condition says that a neuron i will not change its evolution law only because another completely independent neuron j fired (by completely independent, we mean a neuron whose influence $W_{j \rightarrow i}$ is zero).

Note that the values $(W_{i \rightarrow j})_{i \in I}$ are completely irrelevant (they are included in the definition of the model for notational simplicity only), because if $X_n(i) = 1$, we have $U_{n+1}(i) = 0$.

Moreover, note that formally we can't define random variables indexed by \mathbb{R}_+ in the general case because we might have $\lim_{n \rightarrow +\infty} T'_n < +\infty$.

Finally, note that we almost surely never have two neurons firing at the same time (i.e., we have $\forall n \in \mathbb{N}, \sum_{i \in I} X_n(i) \leq 1$ almost surely).

Below, we present a sufficient condition for having $\lim_{n \rightarrow +\infty} T'_n = +\infty$.

Lemma 4.1. If, for every $i \in I$, there exists $t_i > 0$ such that

$$\sup_{u \in \mathbb{R}_+} \int_0^{t_i} \varphi_i(V_i(u, s)) dm(s) < +\infty,$$

then $\lim_{n \rightarrow +\infty} T'_n = +\infty$.

Proof. Note that the condition of the lemma implies that there is a constant $\varepsilon > 0$ such that $\mathbb{P}(\forall i \in I, T_n(i) > t_i) > \varepsilon$ for every $n \in \mathbb{N}$, hence, from the independence, it follows that this event occurs infinitely often in $n \in \mathbb{N}$ almost surely.

Therefore we have that $T_n > \varepsilon$ infinitely often almost surely, hence $\lim_{n \rightarrow +\infty} T'_n = +\infty$. ■

If we have $\lim_{n \rightarrow +\infty} T'_n = +\infty$ almost surely, then we can define further the following random variables.

Let $n_0 = \inf\{n \in \mathbb{N} : T'_n < +\infty\}$ and $T_{-1} = 0$. Furthermore let $X'_{T'_n}(i) = X_n(i)$, for every $i \in I$ and every $n \in \mathbb{N}$ with $n < n_0$; and $X_t(i) = 0$ for every $t \in \mathbb{R}_+ \setminus \{T'_n : n \in \mathbb{N}\}$ and $i \in I$.

Finally, let $U'_t(i) = V_i(U_n(i), t - T'_{n-1})$ for every $t \in \mathbb{R}_+$ such that $T'_{n-1} \leq t < T'_n$ with $n \in \mathbb{N}$ and $n \leq n_0$.

With these definitions (and under these hypothesis), we have stochastic processes $(X'_t)_{t \in \mathbb{R}_+}$ and $(U'_t)_{t \in \mathbb{R}_+}$ on continuous time.

We now present a sufficient condition for the finitude of n_0 , which represents the existence of a last neuron discharge (i.e., the event $\sup\{t \in \mathbb{R}_+ : \exists i \in I, X'_t(i) = 1\} < +\infty$).

Lemma 4.2. If $C > 0$ is such that, for every $u \in \mathbb{R}_+$ and every $i \in I$, we have

$$\int_0^{+\infty} \varphi_i(V_i(u, s)) dm(s) < C,$$

then $n_0 < +\infty$ almost surely.

Proof. It follows directly from the fact that

$$\begin{aligned} \mathbb{P}(\exists n \in \mathbb{N}, T_n = +\infty) &= \mathbb{P}(\exists n \in \mathbb{N}, \forall i \in I, T_n(i) = +\infty) \\ &\geq 1 - \lim_{n \rightarrow \infty} (1 - e^{-C})^n = 1. \end{aligned}$$

■

Remark. Note that the condition of Lemma 4.2 implies the condition of Lemma 4.1.

We now come round to the theorem that characterizes the system's death in the case of non-negative influences.

Theorem 4.3. Suppose $W_{i \rightarrow j} \geq 0$ for every $i, j \in I$, let D be the digraph over I such that $A(D) = \{ij : i \neq j, W_{i \rightarrow j} > 0\}$ and let $S \subset I$ be the set of neurons $i \in I$ such that for every $u \in \mathbb{R}_+$, we have

$$\int_0^{+\infty} \varphi_i(V_i(u, t)) dm(t) < +\infty,$$

and let $R = I \setminus S$.

Suppose also that for every $i \in I$, we have $a(i) > 0$ (i.e., the initial potentials are positive) and $\varphi_i(u) = 0$ if and only if $u = 0$.

Suppose finally that for every $i \in I$, we have $\lim_{t \rightarrow +\infty} V_i(u, t) = 0$ if and only if $u = 0$ or $\lim_{t \rightarrow +\infty} V_i(u', t) = 0$ for every $u' > u$.

Under these circumstances, we have that

$$\mathbb{P}(n_0 < +\infty) > 0$$

if and only if $D[R]$ is a DAG (i.e., the digraph induced by the neurons in R is a directed acyclic graph).

Furthermore, if $\mathbb{P}(n_0 < +\infty) > 0$, then $\mathbb{P}(n_0 < +\infty) = 1$.

Proof. Let's first prove that, for every $i \in R$, we have in fact

$$\int_0^{+\infty} \varphi_i(V_i(u, t)) dm(t) = +\infty,$$

for every $u > 0$ (not only for some u).

We know that for such $i \in R$, there exists $u_0 \in \mathbb{R}_+$ such that

$$\int_0^{+\infty} \varphi_i(V_i(u_0, t)) dm(t) = +\infty,$$

which immediately gives the same for every $u > u_0$ since V_i is non-decreasing on the first coordinate and φ_i is non-decreasing.

Suppose now that $u \in \mathbb{R}_+$ is such that $0 < u < u_0$ and

$$\int_0^{+\infty} \varphi_i(V_i(u, t)) dm(t) < +\infty.$$

Since φ_i only vanishes at 0, the finitude of this integral implies $\lim_{t \rightarrow +\infty} V_i(u, t) = 0$ and since $u > 0$, we have that $\lim_{t \rightarrow +\infty} V_i(u_0, t) = 0$.

This in particular means that there exists $t_0 > 0$ such that $V_i(u_0, t_0) \leq u$, hence, for every $t \in \mathbb{R}_+$, we have

$$V_i(u, t) \geq V_i(V_i(u_0, t_0), t) = V_i(u_0, t_0 + t),$$

where the inequality follows from the fact that V_i is non-decreasing in the first coordinate.

Therefore we have

$$\begin{aligned} \int_0^{+\infty} \varphi_i(V_i(u, t)) dm(t) &\geq \int_0^{+\infty} \varphi_i(V_i(u_0, t_0 + t)) dm(t) \\ &= \int_{t_0}^{+\infty} \varphi_i(V_i(u_0, t)) dm(t) \\ &= \int_0^{+\infty} \varphi_i(V_i(u_0, t)) dm(t) - \int_0^{t_0} \varphi_i(V_i(u_0, t)) dm(t) \\ &= +\infty, \end{aligned}$$

where the last equality follows from the fact that the second integral on the left hand side is finite (because φ_i is non-decreasing, hence locally bounded). But this contradicts the choice of u .

Therefore, for every $i \in R$ and every $u > 0$, we have

$$\int_0^{+\infty} \varphi_i(V_i(u, t)) dm(t) = +\infty.$$

Note finally that this implies that, for every $i \in R$ and every $t, u > 0$, we have $V_i(u, t) > 0$ (because V_i is non-increasing on the second coordinate and φ_i only vanishes at 0).

Suppose now that $\mathbb{P}(n_0 < +\infty) > 0$, and let's prove that $D[R]$ is a DAG.

Suppose not, i.e., suppose that there are neurons $i_1, i_2, \dots, i_k, i_{k+1} \in R$ with $i_1 = i_{k+1}$ and such that $i_j i_{j+1} \in E(D)$ for every $j \in [k]$.

First, let's prove by induction that for every $n \in \mathbb{N}$ there exists $j \in [k]$ such that $U_n(i_j) > 0$ almost surely.

For $n = 0$, this follows immediately from the fact that $a(i) > 0$ for every $i \in I$.

Suppose then that $n > 0$ and that $U_{n-1}(i_j) > 0$.

If $X_n(i_j) = 0$, then we are done, because $U_n(i_j) \geq V_{i_j}(U_{n-1}(i_j), T_n) > 0$ since all neuron influences are non-negative.

Suppose then that $X_n(i_j) = 1$, then we have $X_n(i_{j+1}) = 0$ almost surely, hence $U_n(i_{j+1}) \geq W_{i_j \rightarrow i_{j+1}} > 0$.

Therefore we have

$$\mathbb{P}(\forall n \in \mathbb{N}, \exists j \in [k], U_n(i_j) > 0) = 1.$$

Now, let E denote the event $\{n_0 < +\infty\}$ and let A denote the event $\{n_0 < +\infty; \exists i \in R, U_{n_0}(i) > 0\}$. Note that, since $i_1, i_2, \dots, i_k \in R$, we know that $\mathbb{P}(A | E) = 1$.

On the other hand, for every $i \in R$ and every $n \in \mathbb{N}$, we have

$$\mathbb{P}(T_n(i) < +\infty | U_n(i)) = 1 - \exp\left(-\int_0^{+\infty} \varphi_i(V_i(U_n(i), t)) dm(t)\right)$$

hence $\mathbb{P}(T_n(i) < +\infty \mid U_n(i) > 0) = 1$, because, for every $u > 0$, we have

$$\int_0^{+\infty} \varphi_i(V_i(u, t)) dm(t) = +\infty.$$

Finally, we have

$$\begin{aligned} 0 &= \mathbb{P}(\exists i \in R, T_{n_0}(i) < +\infty \mid E) \\ &\geq \mathbb{P}(\exists i \in R, T_{n_0}(i) < +\infty \mid A \cap E) \mathbb{P}(A \mid E) \\ &= \mathbb{P}(A \mid E), \end{aligned}$$

which is a contradiction.

Therefore $D[R]$ is a DAG.

Suppose now that $D[R]$ is a DAG and let's prove that $\mathbb{P}(n_0 < +\infty) = 1$ (note that we will already prove the final part of the theorem also).

Suppose not, i.e., suppose the event $n_0 = +\infty$ happens with positive probability. From now on, all calculations and statements will be conditioned on the event $\{n_0 = +\infty\}$ and on the event that two neurons never fire at the same time (and this will be omitted from the notation).

The idea is to prove first that there is a state of low potentials that is visited infinitely often. The second step is to prove that we see infinitely often a large sequence of discharges only from neurons of R after reaching a state of low potential. Finally, the third step is to prove that there cannot be such a large sequence of discharges only from neurons of R , which will be a contradiction.

Before we start, let $N = |I|$ and $W = \sum_{i,j \in I} W_{i \rightarrow j}$.

First step.

For every $j \in [N]$ and every $n \in \mathbb{N}$ with $n \geq j-1$, let $A_{j,n} = \{i \in I : U_{n+1}(i) \leq jW\}$ and $E_{j,n}$ denote the event

$$\{|A_{j,n}| \geq j\}.$$

Let's prove by induction on j that $E_{j,n}$ happens infinitely often in n almost surely.

For $j = 1$, note that, since $n_0 = +\infty$, for every $\tilde{n} \in \mathbb{N}$, there exists $n \geq \tilde{n}$ and $i \in I$ such that $X_n(i) = 1$, hence $U_{n+1}(i) = 0$. Therefore $E_{1,n}$ happens infinitely often in n almost surely.

Suppose now that $j \in [N] \setminus \{1\}$ and that $E_{j-1,n}$ happens infinitely often in n almost surely.

Suppose that $E_{j,n}$ does not happen infinitely often in n , then we must have that $E_{j-1,n} \setminus E_{j,n}$ happens infinitely often in n .

On the other hand, note that if there exists $i_0 \in I \setminus A_{j-1,n}$ such that $T_n(i_0) < 1$ and $T_n(i) > 1$ for every $i \in A_{j-1,n}$, then we have $U_{n+1}(i) \leq (j-1)W + W = jW$ for every $i \in A_{j-1,n} \cup \{i_0\}$. This means that, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} &\mathbb{P}(E_{j,n+1} \mid E_{j-1,n} \setminus E_{j,n}) \\ &\geq \mathbb{P}(\exists i_0 \in I \setminus A_{j-1,n}, T_n(i_0) < 1; \forall i \in A_{j-1,n}, T_n(i) > 1 \mid E_{j-1,n} \setminus E_{j,n}) \\ &\geq \left(1 - \max_{i \in I \setminus A_{j-1,n}} \exp\left(-\int_0^1 \varphi_i(V_i((j-1)W, t)) dm(t)\right)\right) \prod_{i \in A_{j-1,n}} \exp\left(-\int_0^1 \varphi_i(V_i((j-1)W, t)) dm(t)\right) \\ &\geq \left(1 - \max_{i \in I} \exp\left(-\int_0^1 \varphi_i(V_i((j-1)W, t)) dm(t)\right)\right) \prod_{i \in I} \exp\left(-\int_0^1 \varphi_i(V_i((j-1)W, t)) dm(t)\right). \end{aligned}$$

Note that the last number is in $(0, 1)$ and is independent of n . Let C be this number. Since $E_{j-1,n} \setminus E_{j,n}$ happens infinitely often in n , we have

$$\mathbb{P}(E_{j,n} \text{ infinitely often in } n) \geq 1 - \lim_{n \rightarrow \infty} (1 - C)^n = 1,$$

which is a contradiction.

Therefore, for every $j \in [N]$, we have that $E_{j,n}$ happens infinitely often in n almost surely.

Second step.

Let $M = 2^{|R|} + 1$ and for every $n \in \mathbb{N}$, let $B_n = \{i \in I : X_n(i) = 1\}$ (remember that we are conditioning on the event $\forall n \in \mathbb{N}, |B_n| = 1$).

Moreover, for every $j \in [M]$ and every $n \in \mathbb{N}$ with $n \geq N + j - 1$, let $F_{j,n}$ denote the event

$$E_{N,n-j+1} \cap \{\forall m \in [j-1], B_{n-m+1} \subset R\},$$

and note that $F_{j+1,n+1} = F_{j,n} \cap \{B_{n+1} \subset R\}$.

Let's prove by induction on j that $F_{j,n}$ happens infinitely often in n almost surely.

For $j = 1$, we have $F_{j,n} = E_{N,n}$ and we already know that $E_{N,n}$ happens infinitely often in n almost surely.

Suppose now that $j \in [M] \setminus \{1\}$ and that $F_{j-1,n}$ happens infinitely often in n almost surely. Note first that, from the definition of $F_{j-1,n}$, we have

$$\mathbb{P}(\forall i \in I, U_{n+1}(i) \leq (N+j)W \mid F_{j-1,n}) = 1.$$

Note that, for every $n \in \mathbb{N}$, we have

$$\mathbb{P}(\forall n \in S, T_n(i) = +\infty \mid F_{j-1,n}) \geq \prod_{i \in S} \left(1 - \exp\left(-\int_0^{+\infty} \varphi_i(V_i((N+j)W, t)) dm(t)\right)\right).$$

Note also that right hand side does not depend on n and is a number in $(0, 1)$ (from the definition of S). Let K be this number. Since $F_{j-1,n}$ happens infinitely often in n , we have

$$\mathbb{P}(F_{j-1,n} \cap \{\forall n \in S, T_n(i) = +\infty\} \text{ infinitely often in } n) \geq 1 - \lim_{n \rightarrow \infty} (1 - K)^n = 1.$$

Note now that

$$\begin{aligned} & \mathbb{P}(T_{n+1} < +\infty \mid F_{j-1,n} \cap \{\forall n \in S, T_n(i) = +\infty\}) \\ &= \mathbb{P}(\exists i \in R, T_{n+1}(i) < +\infty \mid F_{j-1,n} \cap \{\forall n \in S, T_n(i) = +\infty\}) \\ &= \mathbb{P}(F_{j,n} \mid F_{j-1,n} \cap \{\forall n \in S, T_n(i) = +\infty\}), \end{aligned}$$

and since $n_0 = +\infty$, we have that $F_{j,n}$ happens infinitely often in n almost surely.

Therefore, for every $j \in [M]$, we have that $F_{j,n}$ happens infinitely often in n almost surely.

Third step.

Now, let i_1, i_2, \dots, i_k be a topological ordering of the vertices of $D[R]$, i.e., be such that $i_j i_l \in E(D)$ implies $j < l$ (such an ordering always exists in a DAG and can be obtained, for instance, by repeatedly removing one vertex that has indegree 0).

Now let \prec be the strict lexicographic order induced by this order on the power set $\mathcal{P}(R)$ of R and, for every $n \in \mathbb{N}$, let $Q_n = \{i \in R : U_n(i) = 0\}$.

Note that, for every $n \in \mathbb{N}$, we have

$$\mathbb{P}(Q_{n+1} \prec Q_n \mid B_n \subset R) = 1,$$

because if $B_n = \{i\} \subset R$, then $i \in Q_{n+1} \setminus Q_n$ (because i must have a positive potential to fire) and the discharge of i only affects potentials of neurons after i in the topological ordering of $D[R]$.

On the other hand, since $|\mathcal{P}(R)| = 2^R = M - 1$, we know that there cannot be M consecutive occurrences of $B_n \subset R$, because each occurrence take Q_n to a strictly smaller Q_{n+1} . But this is precisely the definition of $F_{M,n}$, which we proved to happen infinitely often in n , so we have a contradiction and the proof is complete. \blacksquare

Observation. Note that the condition $a(i) > 0$ for every $i \in I$ is only important for the first part of the proof (which is expected, since a zero initial potential should work in the direction of yielding $n_0 < +\infty$).

On the other hand, the condition that for every $i \in I$, we have $\lim_{t \rightarrow +\infty} V_i(u, t) = 0$ if and only if $u = 0$ or $\lim_{t \rightarrow +\infty} V_i(u', t) = 0$ for every $u' > u$ might seem artificial at first, but it prevents potential decays that present distinct regimens: one that decays to zero and others that don't.

One example of symptomatic potential decay is

$$V_i(u, t) = (u - \lfloor u \rfloor)e^{-t} + \lfloor u \rfloor,$$

which goes to zero as t goes to $+\infty$ if $u < 1$, but presents a different behaviour for $u \geq 1$.

REFERENCES

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